Note on X-Ray Halos generated by Intergalactic Dust.
E. Commins January 2003 Revised Feb. 21, 2003

1. Introduction

Observations of the apparent magnitudes of scores of distant Type Ia supernovae with redshifts in the range $0.3 < z < 0.85$ have led to the extraordinary discovery that the Hubble expansion is accelerating. The evidence for this claim is that the supernovae are somewhat dimmer at any given redshift than would have been expected for an Einstein de Sitter cosmological model, where $\Omega_m = 1$, $\Omega_\Lambda = 0$.

It has been suggested by Aguirre, however, that the dimming might not be due to cosmic acceleration, but rather to scattering and absorption of supernova light by intervening intergalactic dust. The usual signature for dust is reddening, and there is no evidence for a reddening of Type Ia supernova light that increases with redshift. However, it has been suggested that the intergalactic dust grains might be somewhat larger ($\approx 10^{-4}$ cm in linear size) than ordinary Galactic dust grains ($5 \cdot 10^{-7}$ cm – $2 \cdot 10^{-5}$ cm); hence the intergalactic dust might have "gray" opacity, resulting in much diminished, even unobservable reddening.

Previous constraints on intergalactic dust were too uncertain to rule out this suggestion. Recently however, an observation of X rays in the range 0.3 keV- 8 keV from the distant quasar QSO 1508+5714, made with the Chandra X-ray satellite, was reported by F. Paerels et al of Columbia University. This quasar at z=4.3 is a very intense emitter of X-rays. If intergalactic dust were to exist in the amounts required by Aguirre's hypothesis, one would expect to observe not only the direct X-ray beam from the QSO, but also a surrounding halo with a diameter of order 1 arcmin. This halo would be sufficiently intense that it could have been observed. However, in the expected region around the strong central image, Paerels et al observed no halo, only an extremely weak background. They were thus able to place an upper limit on the intergalactic dust column density, <u>assuming that it consists of spherical grains of size 10^{-4} cm, and that its co-moving density is roughly constant between source and observer.</u> This upper limit is about a factor of 10 less than would be required by Aguirre's suggestion.

The physics of X-ray scattering by dust is straightforward and interesting. The purpose of this short note is to explain in the simplest possible terms how it works, and to derive the relevant formulae that were used by Paerels et al in their very nice work.

2. Derivation of the scattering cross section

Let us begin by considering a spherical dust grain of radius a, with index of refraction $n=n_1+in_2$ where the real (dispersive) part of n is n_1, while the imaginary (absorptive) part is n_2. Suppose a plane electromagnetic wave with frequency ω and wave-number $k=\omega/c$ is incident on the sphere. The general formula describing the scattering of this wave is very complicated for arbitrary values of a, n, and k; (this general case is usually referred to as "Mie" scattering). However, in certain limiting cases, it's possible to treat the scattering in simple and straightforward ways. Fortunately for us, X-ray scattering on micron size dust grains is just such a case.

There are two essential features which make it simple. First of all, the wavelength of an X-ray with energy 1 keV is $\lambda \approx 10^{-7}$ cm. Comparing this with $a=10^{-4}$ cm, we see that

$\lambda << a$; equivalently, $ka >> 1$. Second, X ray scattering is caused by interaction with atomic electrons in the grain. Now since 1 keV >> binding energy of a typical electron in carbon, silicon, or oxygen, the electrons scatter the X rays practically as if they were free. In other words, the scattering is essentially Thompson scattering. Now we can calculate the index of refraction for Thompson scattering by a collection of electrons, and we find that for the densities relevant here, $n_1-1 <<1$, and n_2 also turns out to be exceedingly small. This means that the X-rays penetrate through the grain *almost* as if it were transparent.

Thus we can set up the problem as follows. Imagine the spherical grain to be divided into very small subregions. In each, imagine the electric field of the incident wave, almost unmodified by the presence of the grain. It induces an oscillating electric dipole which radiates a scattered wave, described by the usual Rayleigh scattering amplitude. Now we must sum up all such amplitudes over the volume of the sphere, taking note of the fact that at the distant detector, <u>these amplitudes have different phases.</u> This results in interference between the amplitudes for the various subregions, which turns out to be constructive only within a comparatively narrow forward cone with opening angle $\theta \approx (ka)^{-1}$. For a homogeneous spherical grain, the actual calculation is straightforward and can be done easily. It is in fact exactly the same calculation as in the first Born approximation for scattering of a fast electron by a homogeneous spherical charge distribution, in non-relativistic quantum mechanics. The calculation was first done by Lord Rayleigh in 1885, and then independently by Gans 40 years later (!) and is referred to as "Rayleigh-Gans" scattering.

First we want to calculate the oscillating electric dipole moment in terms of the incident wave electric field E_0 and the index of refraction. Ignoring at first the question of relative phases for different subregions, (and a negligible radiation reaction correction) we can derive the electric dipole moment assuming that the electric field is static. In this case the situation reduces to the very elementary boundary value problem of a dielectric sphere of radius a and dielectric constant $\kappa=n^2$ placed in a uniform electric field E_0. One finds here that the electric field E_i inside the sphere is uniform, parallel to E_0 and of magnitude:

$$E_i = \frac{3}{\kappa + 2} E_0 \qquad (1)$$

Hence the electric displacement is:

$$D = E_i + 4\pi P = \frac{3\kappa}{\kappa + 2} E_0 \qquad (2)$$

so the polarization per unit volume is:

$$P = \frac{3}{4\pi} \frac{\kappa - 1}{\kappa + 2} E_0 \qquad (3)$$

Thus the dipole moment is

$$p = \frac{4\pi}{3} a^3 P = a^3 \frac{\kappa - 1}{\kappa + 2} E_0 \tag{4}$$

If we now consider an oscillating incident electric field, but still ignore phase differences between subregions of the sphere, then $E_0 \to E_0 \exp(i\omega t)$ and the oscillating electric dipole p radiates total power S according to the well-known Larmor formula:

$$S = \frac{p^2 \omega^4}{3c^3} = \frac{1}{3} a^6 \left(\frac{\kappa - 1}{\kappa + 2}\right)^2 \frac{\omega^4 E_0^2}{c^3} \tag{5}$$

The total cross-section is found by dividing S by the incident intensity $\frac{c}{8\pi} E_0^2$:

$$\sigma = \frac{8\pi}{3} k^4 a^6 \left(\frac{\kappa - 1}{\kappa + 2}\right)^2 \tag{6}$$

Now $\kappa = n^2$; hence

$$\begin{aligned} \kappa - 1 &= n^2 - 1 = (n+1)(n-1) \approx 2(n-1) \\ \kappa + 2 &= n^2 + 2 \approx 3 \end{aligned} \tag{7}$$

since n-1<<1. Therefore (6) becomes:

$$\sigma = \frac{32\pi}{27} k^4 a^6 (n-1)^2 \tag{8a}$$

Actually, in (8a) and what follows we should take into account that n could be complex. This can be done simply by the replacement $(n-1)^2 \to |n-1|^2$:

$$\sigma = \frac{32\pi}{27} k^4 a^6 |n-1|^2 \tag{8b}$$

Now we must go back and do the calculation properly by taking into account the phases. For this purpose let's recall equation (4). Making use of the approximations of (7), and writing $V = \frac{4\pi}{3} a^3$, we see that (4) can be expressed as:

$$p = \frac{V}{2\pi} (n-1) E_0 \tag{9}$$

To take account of the phases we now replace (9) by:

$$dp = \frac{n-1}{2\pi} E_0 \bullet e^{i\delta} \, dV \tag{10}$$

for each volume element dV. What is the phase δ ? It's easy to see that:

$$\delta = (\mathbf{k} - \mathbf{k'})\bullet\mathbf{r} \tag{11}$$

where **k**, **k'** are the wave vectors of the incident and scattered waves, respectively, (with |**k'**| = |**k**|), while **r** is a vector from an origin (most conveniently the center of the sphere) to the volume element dV. Fig. 1 shows that:

$$\mathbf{q} = \mathbf{k} - \mathbf{k'} = 2k\ \sin\frac{\theta}{2}\hat{q} \tag{12}$$

where $\hat{q}$ is a unit vector in the direction of **k**-**k'**, and θ is the scattering angle.

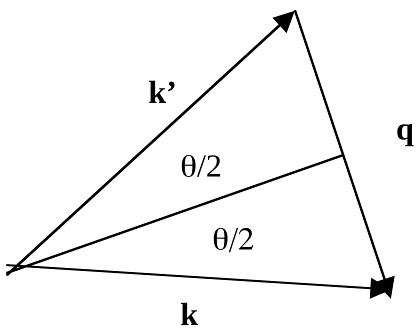


Fig.1

We can now calculate the total dipole moment by integrating over the sphere using spherical polar coordinates: r, polar angle β and azimuthal angle ϕ, where the polar axis is chosen to be parallel to **q**. We have:

$$\begin{aligned} p &= \frac{n-1}{2\pi}E_0 \bullet 2\pi\int_0^a r^2 dr\int_0^\pi \sin\beta \bullet \exp\left[2ik\sin\frac{\theta}{2}r\cos\beta\right]d\beta \\ &= (n\text{-}1)E_0\int_0^a r^2 dr\frac{\exp\left(2ik\sin\frac{\theta}{2}r\right)-\exp\left(-2ik\sin\frac{\theta}{2}r\right)}{2ik\sin\frac{\theta}{2}r} \\ &= \frac{2a^3(n-1)E_0}{u}\int_0^a r\ \sin\left(\frac{ur}{a}\right)dr \end{aligned} \tag{13}$$

where u=2ka sin(θ/2). We easily evaluate the last integral to obtain:

$$p = \frac{2}{3}a^3(n-1)E_0\left[3\frac{\sin(u)-u\bullet\cos(u)}{u^3}\right] \tag{14}$$

We see that our new expression for p, equation (14), is the same as the old one (eq'n (9)), except for the factor in brackets, which is unity for u=0 (when θ=0), but drops rapidly when u approaches unity (that is, when $\theta \approx (ka)^{-1}$).

If the polarization of the incident wave is in the plane of scattering, the electric field of the scattered wave in the radiation zone is:

$$E_{scat} = k^2 p \frac{\cos\theta}{r} e^{ikr} \tag{15}$$

On the other hand, if the polarization of the incident wave is perpendicular to the plane of scattering we have:

$$E_{scat} = k^2 p \frac{e^{ikr}}{r} \tag{16}$$

Thus the differential cross section for scattering, averaged over incident polarizations, is:

$$\frac{d\sigma}{d\Omega} = \frac{4}{9}|n-1|^2 a^6 k^4 \frac{1+\cos^2\theta}{2} G \tag{17}$$

where $G = \left[3\frac{\sin(u) - u\bullet\cos(u)}{u^3}\right]^2$. Now it turns out that an excellent approximation to G is $G \approx \exp\left[-\frac{2}{9}(ka)^2\theta^2\right]$. (See Fig.2). Using this in (17) and recalling that the differential cross section is negligible for $\theta >> (ka)^{-1}$ so that we can replace $\cos^2\theta$ by 1, we calculate the total cross section as follows:

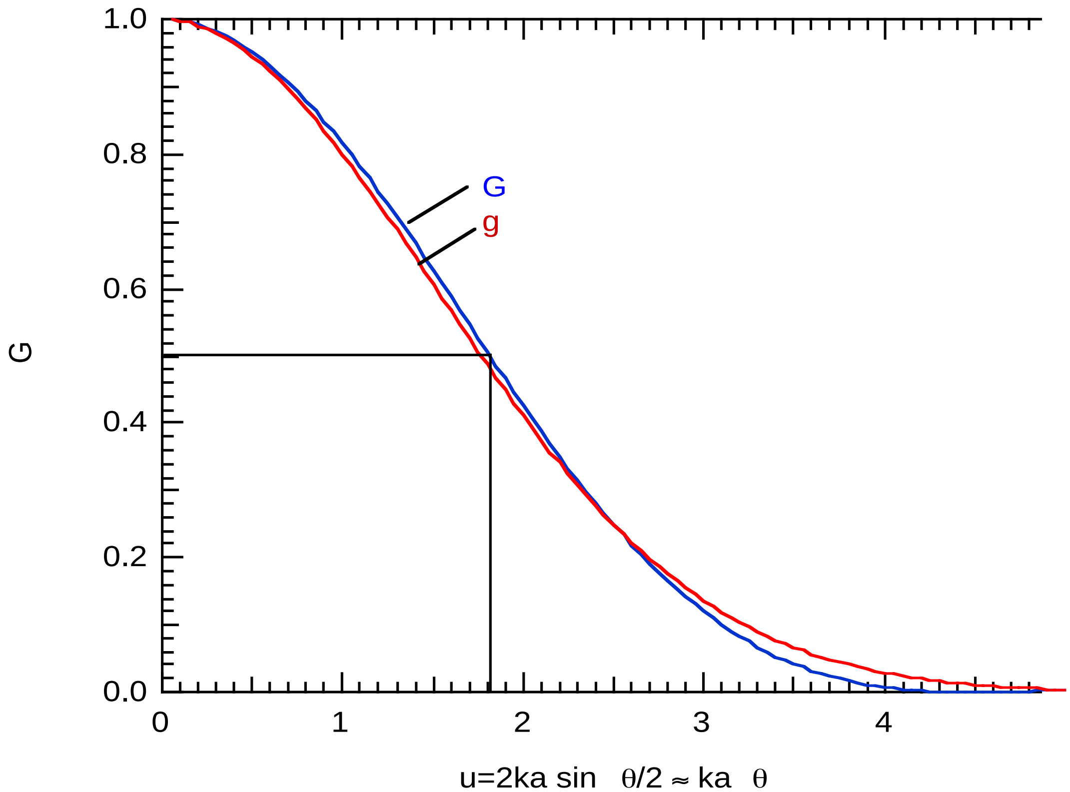
$$\begin{aligned}
\sigma &= \frac{4}{9}|n-1|^2 a^6 k^4 \bullet 2\pi\int_0^\pi \sin\theta \bullet \exp\left(-\frac{2}{9}k^2a^2\theta^2\right)d\theta \\
&\approx \frac{8\pi}{9}|n-1|^2 a^6 k^4 \int_0^\infty \theta \exp\left(-\frac{2}{9}k^2a^2\theta^2\right)\mathrm{d}\theta \\
&= 2\pi\,|n-1|^2\,\mathrm{k}^2\mathrm{a}^4
\end{aligned} \tag{18}$$

Now let's consider the index of refraction n. As we mentioned, the electrons can be treated as nearly free, so that the scattering is essentially Thompson scattering, as far as n_1 is concerned. Then, it is easy to show that:

$$n_1 = 1 - 2\pi n_e r_0 \frac{1}{k^2} \tag{19}$$

where n_e is the number of electrons per cm^3 and $r_0 = \frac{e^2}{m_e c^2} = 2.8\bullet 10^{-13}$ cm is the classical radius of the electron. As for n_2, it is proportional to the X-ray absorption coefficient, which is negligible below the various photo-electric absorption K edges of carbon, silicon, oxygen; and drops rapidly above those edges. Altogether, the effect of n_2 turns out to be very small, although not completely negligible. Using (19) as a good initial approximation in (18), we find:

$$\sigma = 8\pi^3 n_e^2 \frac{a^4 r_0^2}{k^2} \tag{20}$$



$$G = \left[\frac{3}{u^3}(\sin u - u\cos u)\right]^2$$

$$g = \exp\left[-\frac{2}{9}u^2\right]$$

Fig.2 Comparison of the functions G and g. The black lines indicate that the half width at half maximum for each function occurs at u≈1.8.

Eq'n (20), which is essentially correct, shows that the scattering cross section is proportional to the grain electron density squared, the grain radius to the 4th power, and the X-ray energy to the (-2) power. We can still make some minor improvements in the formula, by expressing the electron density in terms of the grain mass density ρ in 3 g cm^{-3} (a typical grain mass density), the radius in microns, and the X-ray energy in keV. We also refine the effect of the atomic electrons slightly by including an electron atomic form factor F(E) (which in zeroth approximation is just the atomic number Z but does actually depart slightly from it near the K absorption edges because of the effect of n_2). Thus we find:

$$\sigma_{scat} = 6.3 \bullet 10^{-7} \left(\frac{2Z}{A}\right)^2 \left(\frac{\rho}{3 \text{ g cm}^{-3}}\right)^2 \left(\frac{a}{micron}\right)^4 \left(\frac{E_g}{keV}\right)^{-2} \left[\frac{F(E_g)}{Z}\right]^2 \text{ cm}^2 \qquad (21)$$

In (21) we have written a subscript g on the energy to indicate that it is the X ray energy that would be observed in the grain frame of reference. It is very important for what follows to note that the X ray energy observed by the Chandra satellite: E, is related to E_g by:

$$E = \frac{E_g}{1+z}$$

where z is the redshift of the grain. Thus we rewrite (21) as follows:

$$\sigma_{scat} = 6.3 \bullet 10^{-7} K(a,\rho,Z,A) \frac{1}{E^2(1+z)^2} \qquad (22)$$

where

$$K = \left(\frac{2Z}{A}\right)^2 \left(\frac{\rho}{3}\right)^2 \left(\frac{a}{micron}\right)^4 \left(\frac{F(E_g)}{Z}\right)^2 \qquad (23)$$

3. The Halo Intensity

Now that we have a formula for the scattering cross section we can think about the detected X ray image. It should consist of a point source core of unscattered photons, and a surrounding halo of scattered photons. Let the intensity of the core without dust attenuation be I. Then taking into account the dust we have:

$$I_{core} = I \bullet \exp\left(-\int \sigma_{scat} n_g dl\right)$$
$$I_{halo} = I\left[1 - \exp\left(-\int \sigma_{scat} n_g dl\right)\right]$$

where n_g is the dust grain number density, and l is the distance along the line of sight. Hence:

$$\frac{I_{halo}}{I_{halo} + I_{core}} = 1 - \exp\left(-\int \sigma_{scat} n_g dl\right) \approx \int \sigma_{scat} n_g dl = \tau \qquad (24a)$$

where τ is the optical depth. Eq'n (24) is valid if the optical depth is sufficiently small (τ<<1) so that multiple scattering does not have a high probability. In this case:

$$\tau \approx \frac{I_{halo}}{I_{core}} \qquad (24b)$$

is a good approximation. (If $\tau \approx 1$ a complicated radiative transfer calculation would be necessary.)

We now consider the optical depth in some detail, taking into account some features of General Relativity. Our starting point is the formula for the differential optical depth:

$$d\tau = \sigma_{scat} n_g dl \qquad (25)$$

We shall also need to make use of the quantity

$$a = \frac{R}{R_0} = \frac{1}{1+z} \qquad (26)$$

where R is the scale parameter of the universe at some earlier time characterized by the redshift z, while R_0 is the present value of the scale parameter. (The symbol a used here should not be confused with the same symbol used for the grain radius!) We shall also need Friedmann's second equation, which is:

$$\frac{da}{dT} = \sqrt{\Omega_m\left(\frac{1}{a} - 1\right) + \Omega_\Lambda\left(a^2 - 1\right) + 1} \qquad (27)$$

where $T=H_0(t-t_0)$, and H_0 is the Hubble constant, while t_0 is the present time and t is any time, past, present, or future; as described by a co-moving observer. According to the standard cosmological model, equation (27) describes the Hubble expansion at any epoch long enough after the radiation era that we can neglect $\Omega_{radiation}$ in comparison to Ω_m and/or Ω_Λ. For a flat universe, which is strongly favored by the results of measurements of the CMB fluctuation spectrum, we have $\Omega_m + \Omega_\Lambda = 1$, and (27) becomes:

$$\frac{da}{dT} = \sqrt{\Omega_m \frac{1}{a} + \Omega_\Lambda a^2} \qquad (28)$$

Since the evidence for a flat universe is quite strong, we shall assume the validity of (28) in the following discussion.

Now we wish to convert the quantity dl appearing in (25) into a more useful form, by means of the preceding equations from General Relativity. First, we note that for a photon:

$$dl = cdt = \frac{c}{H_0} dT \tag{29}$$

Furthermore, from (26) we have:

$$da = -\frac{dz}{(1+z)^2} \tag{30}$$

Thus, making use of (26) and (28-30) we arrive at:

$$dl = \frac{c}{H_0} \frac{dz}{(1+z)} \frac{1}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}} \tag{31}$$

where the sign is positive in (31) because we now adopt the convention that l increases as we go from observer to source.

Let us define the "co-moving" grain number density $n_{00}(z)$ by the relation:

$$n_{00}(z) = n_0 (1+z)^{-3} \tag{32}$$

Then the differential optical depth of eq'n (25) can be expressed as:

$$d\tau = \frac{c}{H_0} n_{00}(z) \frac{(1+z)^2}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}} \sigma dz \tag{33}$$

Substituting expression (22) for the cross section σ in (33), we arrive at:

$$d\tau = 6.3 \bullet 10^{-7} \frac{c}{H_0} n_{00}(z) \frac{dz}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}} \frac{K}{E^2} \tag{34}$$

where we remind ourselves that E is the X-ray energy in keV measured by us.

For any given E, and choice of $n_{00}(z)$, (34) can be integrated to yield the optical depth. Let us suppose at first that n_{00} is constant all the way to z=4.3. Now n_{00} is the number of dust grains per cm^3 in a co-moving volume. We may express this in terms of ρ', the co-moving mass density of dust grains, as follows:

$$\rho' = n_{00} \bullet \frac{4\pi}{3} \rho a^3 \tag{35}$$

where, as we have previously noted, ρ is the mass density of grain material, and a is the grain radius. We introduce Ω_d, the ratio of ρ' to the critical mass density. Then,

$$\rho' = \frac{3H_0^2}{8\pi G}\Omega_d$$

hence:

$$n_{00} = \frac{9}{32\pi^2}\frac{H_0^2}{G}\Omega_d\frac{1}{\rho a^3} \tag{36}$$

Inserting (36) in (34), and assuming H_0=70 km/s-Mpc, we obtain:

$$d\tau = .062\left(\frac{\Omega_d}{10^{-5}}\right)\left(\frac{2Z}{A}\right)^2\left(\frac{F(E_g)}{Z}\right)^2\left(\frac{\rho}{3}\right)\left(\frac{a}{micron}\right)\frac{1}{E^2}\frac{dz}{\sqrt{\Omega_m(1+z)^3+\Omega_\Lambda}} \tag{37}$$

Of course Aguirre's hypothesis does <u>not</u> require the co-moving dust density to be constant out to z=4.3. The current supernovae results could be explained by intergalactic gray dust if, for example, the co-moving density was constant out to a certain z_c (for example z_c=0.5,) and then was zero for larger values of z. We can take this admittedly artificial possibility into account by integrating (37) from z=0 to that "cutoff" value z_c to obtain the total optical depth.

To illustrate this procedure, let's assume that in (37) $\Omega_d=10^{-5}$, 2Z=A, $F(E_g)$ =Z, ρ=3 g cm^{-3}, and a =1 micron. Also, suppose Ω_m=1 and Ω_Λ=0. Then (37) is easily integrated for any given E to yield:

$$\tau(E) = .124 \bullet E^{-2}\left[1-\frac{1}{\left(1+z_c\right)^{1/2}}\right] \tag{38}$$

Now, from (24b), we have:

$$I_{core}(E)\tau(E) = I_{halo}(E) \tag{39}$$

The energy spectrum of the core is given by $I_{core}(E)=AE^{-p}$ where A is a constant, and p=.5 to a good approximation. Hence,

$$I_{halo}(E) = .124A\left[1-\frac{1}{\left(1+z_c\right)^{1/2}}\right]E^{-5/2} \tag{40}$$

We integrate this expression over the observed band of energies (.3 to 8 keV in the Paerels experiment) to obtain the total halo intensity:

$$I_{halo} = .499A\left[1-\frac{1}{(1+z_c)^{1/2}}\right] \tag{41}$$

Meanwhile the total core intensity is $I_{core} = A\int_{.3}^{8}E^{-1/2}dE = 4.56A$. Therefore,

$$\frac{I_{halo}}{I_{core}} = .109\left[1 - \frac{1}{(1+z_c)^{1/2}}\right] \tag{42}$$

Returning to the general case, for given X-ray energy, dust grain radius, etc, we would like to know in general how the optical depth varies as a function of the cutoff redshift z_c. From (37), we can see that:

$$\chi \equiv \frac{\tau(z_c)}{\tau(z_c = 4.3)} = \frac{\int_0^{z_c} \frac{dx}{\sqrt{\Omega_m(1+x)^3 + \Omega_\Lambda}}}{\int_0^{4.3} \frac{dx}{\sqrt{\Omega_m(1+x)^3 + \Omega_\Lambda}}} \tag{43}$$

This quantity is plotted versus z_c in Fig.3 for $(\Omega_m, \Omega_\Lambda)$=(1,0) and (.3, .7). We can see from the figure that there is not much difference between the two curves, and that $\chi \approx .3$ for z_c =.5, while $\chi \approx .5$ for z_c=1.

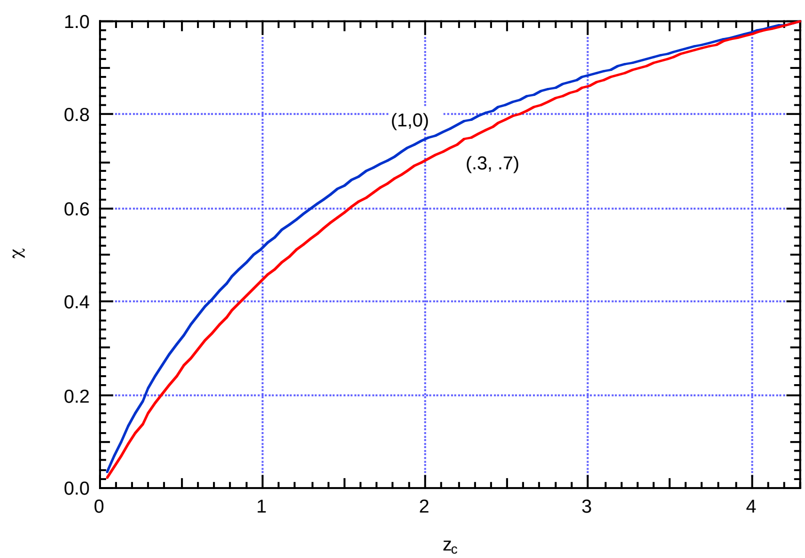


Fig. 3. The ratio χ plotted versus z_c.

4. Angular distribution of Halo radiation

In what follows, the subscripts 1 and 2 refer to the dust grain and the QSO source, respectively (see Fig. 4). Suppose that the source emits $L(\nu_2)$ ergs s^{-1} Hz^{-1} at X ray frequency ν_2 in the QSO frame. Then the flux of radiation at the dust grain is given by:

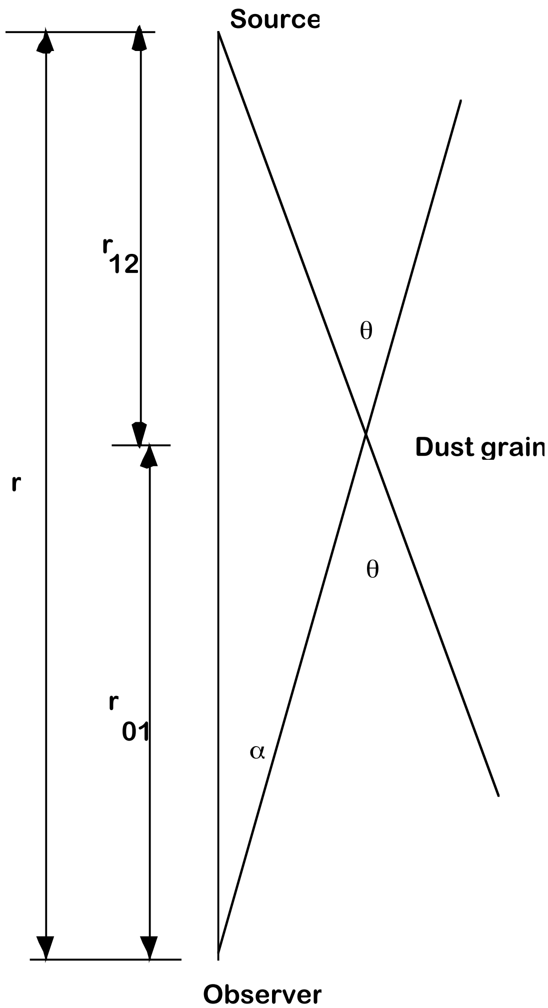


Fig.4 Diagram showing the relation between observed halo angle α and scattering angle θ. Actual angles are extremely small.

$$F(\nu_1) = \frac{L(\nu_2)}{4\pi R_1^2 r_{12}^2}\frac{1}{(1+z')}\text{erg s}^{-1}\text{ Hz}^{-1}\text{ cm}^{-2} \qquad (44)$$

where ν_1 is the X ray frequency, R_1 is the scale parameter, and z' is the redshift of the QSO, all with respect to the dust grain. Also, assuming that the curvature index is zero (flat universe), r_{12} is given by the formula:

$$r_{12} = \frac{c}{H_0 R_0}\int_{z_1}^{z_2}\frac{dz}{\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}} \qquad (45)$$

(For derivations of (44,45) see for example Weinberg's Gravitation and Cosmology, Part V, Secs2,3,4).

We shall assume that the angles θ and α are extremely small, so that sin α≈ tanα≈α and sinθ ≈ tanθ ≈θ. Then

$$\alpha\, r = \theta\, r_{12}$$

From this equation, defining $x=r_{01}/r$, we obtain:

$$\alpha = \theta\,(1 - x) \tag{46}$$

Now, $\frac{R_2}{R_1} = \frac{1}{1+z'}$, $\frac{R_2}{R_0} = \frac{1}{1+z_2}$, and $\frac{R_1}{R_0} = \frac{1}{1+z_1}$. Therefore,

$$\frac{1}{1+z'} = \frac{1+z_1}{1+z_2}$$

Hence (44) becomes:

$$F(\nu_1) = \frac{L(\nu_2)}{4\pi R_0^2 r^2}\frac{(1+z_1)^3}{(1+z_2)}\frac{1}{(1-\mathrm{x})^2}\,\text{erg s}^{-1}\text{ Hz}^{-1}\text{ cm}^{-2} \tag{47}$$

The <u>flux of photons</u> $f(\nu_1)$ at the grain (in photons per s per Hz per cm^2) is obtained from $F(\nu_1)$ by dividing by $h\nu_1=E_0(1+z_1)$:

$$f(\nu_1) = \frac{L(\nu_2)}{4\pi R_0^2 r^2 E}\frac{(1+z_1)^2}{(1+z_2)}\frac{1}{(1-\mathrm{x})^2}\,\text{photons s}^{-1}\text{ Hz}^{-1}\text{ cm}^{-2} \tag{48}$$

Now consider a small patch of area dA_0 at the detector. Then, since α, θ are very small, the solid angle subtended by dA_0 at the grain is:

$$d\Omega = \frac{dA_0}{R_0^2 r_{01}^2}. \tag{49}$$

On the other hand we can consider a small right circular cylinder at the grain, of length dl, cross sectional area dA, and axis along the X ray path from source to grain. Then, since α, θ are very small the solid angle subtended by this cylinder at the observer is

$$d\Omega_0 = \alpha\, \mathrm{d}\alpha\, d\phi = \frac{dA}{R_1^2 r_{01}^2} = \frac{(1+z_1)^2\, dA}{R_0^2 r_{01}^2} \tag{50}$$

Note that in (49) and (50) we have used some geometric relations from General Relativity; (see the previous reference to Weinberg, for example).

In the grain frame, the number of X rays scattered from the small cylinder per second per Hz into the solid angle dΩ is:

$$dN = f(\nu_1) n_0 dl dA d\Omega \frac{\partial\sigma}{\partial\Omega} \tag{51}$$

Now from (49), (50) we have $dA d\Omega = \alpha\, d\alpha\, d\phi\, dA_0 (1+z_1)^{-2}$, and also, $n_0 = n_{00}(1+z_1)^3$. Making these substitutions in (51) and also employing (48), we obtain:

$$dN = \frac{L(\nu_2)}{4\pi R_0^2 r^2 E} \frac{(1+z_1)^3}{\left(1+z_2\right)} \frac{n_{00}}{(1-x)^2}\, dl \frac{\partial\sigma}{\partial\Omega} \alpha\, d\alpha\, d\phi dA_0 \tag{52}$$

Next, we divide both sides of (52) by dA_0 and multiply by E to obtain the intensity per Hz at the detector in the solid angle α dαdϕ from the length element dl at redshift z_1:

$$dI = \frac{L(\nu_2)}{4\pi R_0^2 r^2} \frac{(1+z_1)^3}{\left(1+z_2\right)} \frac{n_{00}}{(1-x)^2}\, dl \frac{\partial\sigma}{\partial\Omega} \alpha\, d\alpha\, d\phi \tag{53}$$

Now, as was shown in (31), we can express dl in terms of z_1 as follows:

$$dl = \frac{c}{H_0} \frac{dz_1}{\left(1+z_1\right)} \frac{1}{\sqrt{\Omega_m (1+z_1)^3 + \Omega_\Lambda}} \tag{54}$$

Also, we recall from (17) that to a good approximation, the differential cross section may be written as:

$$\frac{\partial\sigma}{\partial\Omega} = \left[\frac{16\pi^2}{9} n_e^2 r_0^2 a^6\right] \bullet \exp\left(-\frac{2}{9} a^2 k^2 \theta^2\right)$$

The quantity in square brackets is just a constant that we will label as q. In the exponent, k is the X ray wave number as seen by the grain:

$$k = \frac{E_1}{\hbar c} = \frac{E(1+z_1)}{\hbar c}$$

and also $\theta = \dfrac{\alpha}{(1-x)}$. Hence,

$$\frac{\partial\sigma}{\partial\Omega} = q \bullet \exp\left[-\frac{2}{9} a^2 \frac{E^2\left(1+z_1\right)^2}{\hbar^2 c^2} \frac{\alpha^2}{(1-x)^2}\right] \tag{55}$$

Inserting (54) and (55) in (53) we finally obtain:

$$dI = \frac{c}{H_0} q \frac{L(\nu_2)}{4\pi R_0^2 r^2} \frac{n_{00}}{\left(1+z_2\right)} \frac{(1+z_1)^2 dz_1}{\sqrt{\Omega_m \left(1+z_1\right)^3 + \Omega_\Lambda}} \frac{1}{(1-x)^2} \exp\left[-\frac{2a^2}{9\hbar^2 c^2} \frac{E^2 (1+z_1)^2 \alpha^2}{(1-x)^2}\right] \alpha\, \mathrm{d}\alpha\, d\phi \tag{56}$$

Integrating this expression over z_1 from 0 to the cutoff redshift z_c, and also integrating over the QSO energy spectrum, we obtain an intensity per unit solid angle, that is plotted versus α in Fig. 5. The logarithmic vertical scale contains an arbitrary additive constant, but is the same for each cutoff redshift z_c.

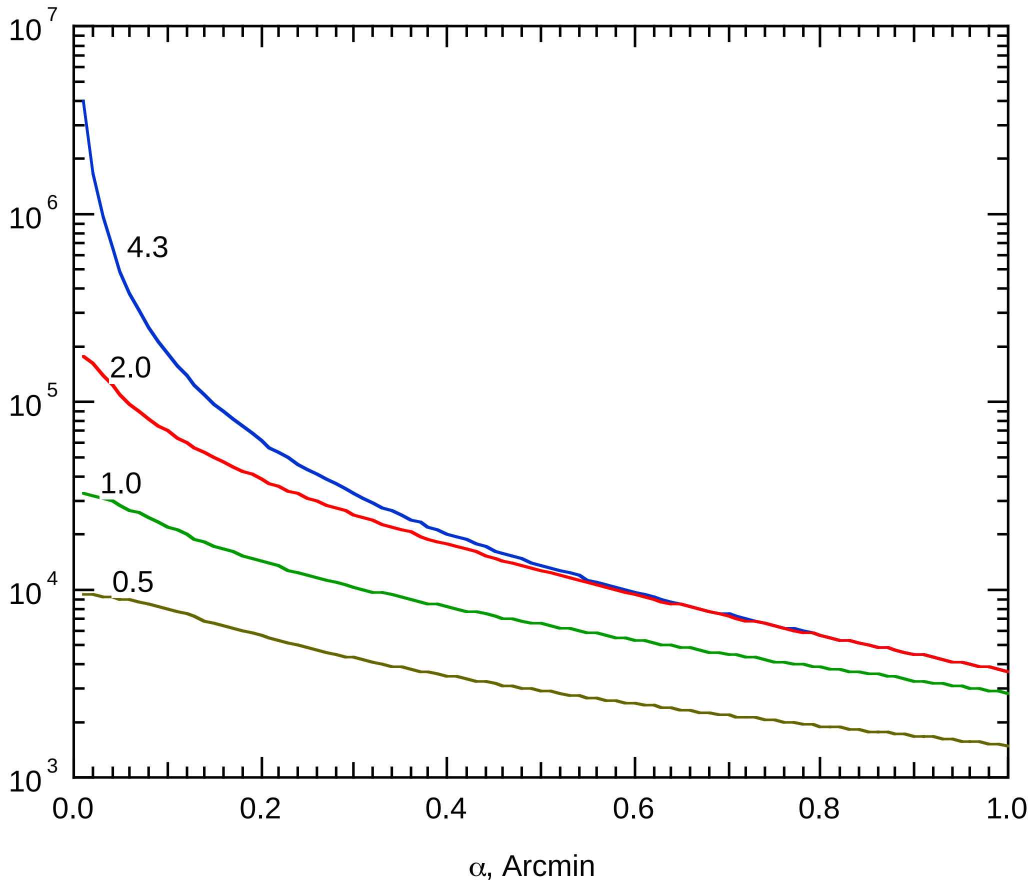


Fig. 5. Intensity per unit solid angle plotted versus angle α(in arcmin) for Ω_m=1, Ω_Λ=0. The ordinate scale is arbitrary, but the same for all cutoff redshifts (z_c = 0.5, 1.0, 2.0, 4.3). The results for Ω_m=0.3, Ω_Λ=0.7 are quite similar.

5. Cylindrical Grains

Aguirre has argued that the hypothetical grey dust grains are probably elongated "needles" rather than spheres.To take this possibility into account, we re-derive the Rayleigh-Gans scattering cross section for cylindrical grains, by recalling eq'ns (10-12): The dipole moment induced in a small volume dV of the grain by the incident X-ray electric field is:

$$dp = \frac{n-1}{2\pi} E_0 \bullet e^{i\delta} dV \qquad (10)$$

The phase shift δ is:

$$\delta = (\mathbf{k} - \mathbf{k}')\bullet\mathbf{r} \qquad (11)$$

where **r** is the position vector of the volume element dV relative to a convenient origin, and:

$$\mathbf{q} = \mathbf{k} - \mathbf{k'} = 2k\,\sin\frac{\theta}{2}\hat{q} \qquad (12)$$

where θ is the scattering angle. To carry out our calculation of the total dipole moment for a cylinder, we choose the geometry illustrated in Fig.6. Let the cylinder have radius a

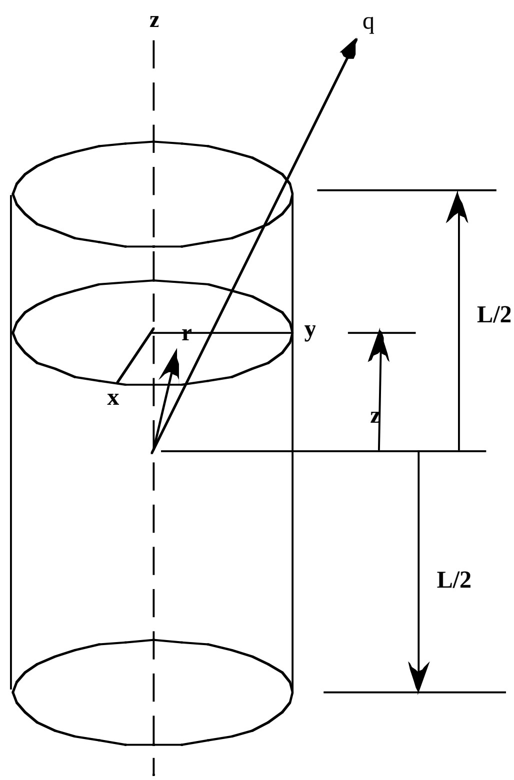


Fig. 6.

and length L. Locate the origin of coordinates O at the center of the cylinder. Suppose vector **q** makes angle β with respect to the cylinder axis. Choose x and y coordinates so that **q** lies in the yz plane. Let **r** be the position vector to a volume element with x,y,z coordinate values x,y,z (in the disk that is shown, with thickness dz). Then:

$$\begin{aligned}\delta &= \vec{q} \bullet \vec{r} = q_y y + q_z z \\ &= 2\mathrm{ksin}\frac{\theta}{2}\left[\mathrm{ysin}\beta + \mathrm{zcos}\beta\right]\end{aligned} \qquad (57)$$

Now divide the disk at z into strips running parallel to the x axis, of width dy. The volume of the strip at y is:

$$dV = 2\sqrt{a^2 - y^2}\,dydz \qquad (58)$$

and the phase shift δ is constant in each strip. Thus the total dipole moment is:

$$p = \frac{n-1}{2\pi} E_0 \int_{-L/2}^{L/2} \exp\left[2ik \sin\frac{\theta}{2}\cos\beta \bullet z\right] dz \int_{-a}^{a} 2\sqrt{a^2 - y^2} \exp\left[2ik \sin\frac{\theta}{2}\sin\beta \bullet y\right] dy \tag{59}$$

Let's define $u = Lk \sin\frac{\theta}{2}\cos\beta$ and consider first the integral over z. Making the substitution t=2z/L, we see that this integral can be written as:

$$L\int_0^1 \cos(ut)dt = L\frac{\sin(u)}{u} \tag{60}$$

Next, consider the integral over y. Defining $v = 2ka\sin\frac{\theta}{2}\sin\beta$ and s=y/a, we see that this integral can be expressed as:

$$4a^2\int_0^1 \sqrt{1-s^2}\cos(vs)ds = 2\pi a^2 \frac{J_1(v)}{v} \tag{61}$$

where J_1 is the Bessel function of order 1. Hence the dipole moment is:

$$p = \frac{n-1}{2\pi} VE_0 \bullet \frac{\sin(u)}{u} \bullet \frac{2J_1(v)}{v} \tag{62}$$

From this expression, and using methods similar to those employed for spherical grains we can derive the differential cross section:

$$\frac{\partial\sigma}{\partial\Omega} = .624\left(\frac{2Z}{A}\right)^2 \left(\frac{\rho}{3}\right)^2 \left(\frac{a}{micron}\right)^4 \left(\frac{L}{micron}\right)^2 \frac{\sin^2 u}{u^2}\left(\frac{2J_1(v)}{v}\right)^2 \tag{63}$$

The function $S(v)=[2J_1(v)/v]^2$ appears in this differential cross section. It is plotted in Fig. 7, (blue curve) together with a good approximation to it:
$s(v)=\exp(-.2801v^2)$, (the red curve).

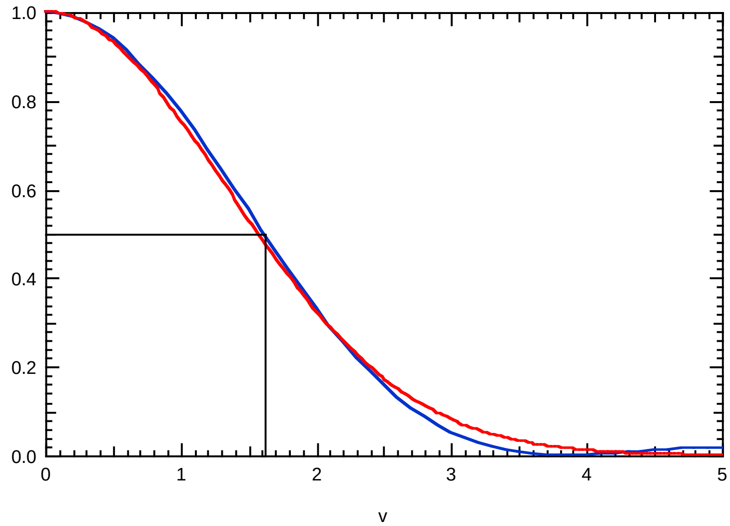


Fig. 7 The function S(v): blue curve, and the function s(v):red curve, are plotted versus v. The half-width at half maximum for each function appears at v≈1.6.

In Fig. 8. we plot the function $F(u)=[\sin u/u]^2$: (blue curve) which also appears in the differential cross section, together the useful approximation to it: $f(u)=\exp(-.3813u^2)$: (red curve).

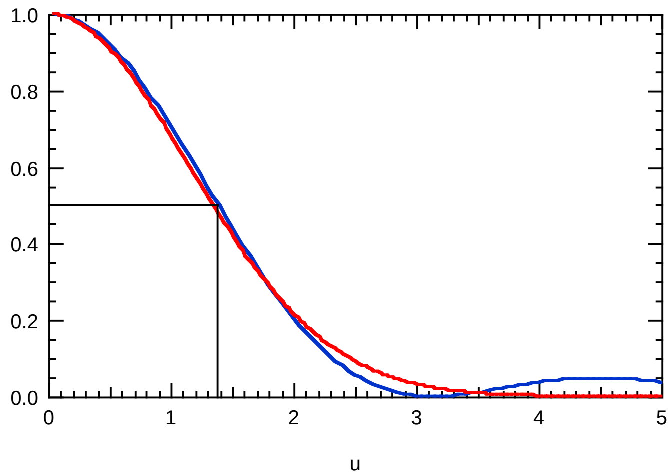


Fig.8 The function F(u) (blue curve) and the function f(u) (red curve) are plotted versus u. The half width at half maximum for each function appears at u≈1.4.

Making use of the approximations shown in Figs 7, 8, and assuming that θ is always very small so that $\sin\frac{\theta}{2} \approx \frac{\theta}{2}$, we average the differential cross section over random cylinder orientations as follows:

$$\left\langle\frac{\partial\sigma}{\partial\Omega}\right\rangle = .0195\left(\frac{2Z}{A}\right)^2\left(\frac{\rho}{3}\right)^2\left(\frac{d}{micron}\right)^4\left(\frac{L}{micron}\right)^2 \bullet$$
$$\int_0^\pi \exp\left[-A\left(k\frac{L}{2}\theta\cos\beta\right)^2 - B\left(k\frac{d}{2}\theta\sin\beta\right)^2\right]\sin\beta d\beta \qquad (64)$$

where A=.381, B=.2801, and d=2a is the cylinder diameter. The integral in (64) can be written as follows:

$$Int = \exp\left[-\frac{1}{4}Bk^2d^2\theta^2\right]\int_0^\pi \exp\left[-\frac{1}{4}Ak^2L^2\theta^2\left(1-\frac{B}{A}\left(\frac{d}{L}\right)^2\right)\cos^2\beta\right]\sin\beta d\beta \qquad (65)$$

Let $q_0 = \frac{1}{4}Ak^2L^2\theta^2\left(1-\frac{B}{A}\left(\frac{d}{L}\right)^2\right)$ and $y=q_0^{1/2}\cos\beta$. Then, (65) becomes:

$$Int = \frac{\exp\left[-\frac{1}{4}Bk^2d^2\theta^2\right]}{q_0^{1/2}}\int_0^{q_0^{1/2}} e^{-y^2}dy$$
$$= \frac{\sqrt{\pi}}{2q_0^{1/2}}\exp\left[-\frac{1}{4}Bk^2d^2\theta^2\right]erf\left(q_0^{1/2}\right) \qquad (66)$$

We now make use of eq'ns (53, 54, 64, and 66) to write the halo intensity received per Hz in solid angle $\alpha\ d\alpha\ d\phi$ in terms of the core flux per Hz $F(\nu_2)$:

$$dI = \left[F\left(\nu_2\right)\right]\frac{(1+z_1)^2}{\left(1+z_2\right)}\frac{n_{00}}{(1-x)^2}\frac{c}{H_0}\frac{dz_1}{\sqrt{\Omega_m\left(1+z_1\right)^3+\Omega_\Lambda}} \bullet$$
$$.0195\left(\frac{2Z}{A}\right)^2\left(\frac{\rho}{3}\right)^2\left(\frac{d}{micron}\right)^4\left(\frac{L}{micron}\right)^2 \bullet \qquad (67)$$
$$\frac{\sqrt{\pi}}{2q_0^{1/2}}\exp\left[-\frac{1}{4}Bk^2d^2\theta^2\right]erf\left(q_0^{1/2}\right)\alpha\, d\alpha\, d\phi$$

Now as was previously shown (see eq'n 36), the co-moving density n_{00} can be expressed as:

$$n_{00} = \frac{3H_0^2}{8\pi G}\Omega_d\frac{1}{\rho V}$$

which, for a cylinder with volume $V=\pi d^2L/4$, becomes:

$$n_{00} = \frac{1}{2\pi^2}\frac{H_0^2}{G}\Omega_d\frac{1}{d^2L}\left(\frac{3}{\rho}\right) \qquad (68)$$

Making this substitution in (67) and inserting the numerical value of cH_0/G we obtain:

$$dI = 9.02 \bullet 10^8 \left[F(\nu_2)\right] \frac{(1+z_1)^2}{(1+z_2)} \frac{1}{(1-x)^2} \frac{dz_1}{\sqrt{\Omega_m (1+z_1)^3 + \Omega_\Lambda}} \bullet$$

$$\left(\frac{2Z}{A}\right)^2 \left(\frac{\rho}{3}\right) \left(\frac{d}{micron}\right)^2 \left(\frac{L}{micron}\right) \Omega_d \bullet \tag{69}$$

$$\frac{1}{q_0^{1/2}} \exp\left[-\frac{1}{4} Bk^2 d^2 \theta^2\right] erf\left(q_0^{1/2}\right) \alpha \, d\alpha \, d\phi$$

Let's now calculate the optical depth. For this purpose we must integrate (69) over all angles:

$$I = 9.02 \bullet 10^8 \left[F(\nu_2)\right] \frac{(1+z_1)^2}{(1+z_2)} \frac{1}{(1-x)^2} \frac{dz_1}{\sqrt{\Omega_m (1+z_1)^3 + \Omega_\Lambda}} \bullet$$

$$\left(\frac{2Z}{A}\right)^2 \left(\frac{\rho}{3}\right) \left(\frac{d}{micron}\right)^2 \left(\frac{L}{micron}\right) \Omega_d \bullet \tag{70}$$

$$2\pi \int_0^\infty \frac{1}{q_0^{1/2}} \exp\left[-\frac{1}{4} Bk^2 d^2 \theta^2\right] erf\left(q_0^{1/2}\right) \alpha \, d\alpha$$

Now, $k=5.06 \bullet 10^7 (1+z_1)E$, where E is the observed X ray energy in keV. Also, $\theta = \alpha/(1-x)$; (recall Fig. 4). Therefore, $q_0^{1/2}=k_1\alpha$, where

$$k_1 \equiv 1.561 \bullet 10^3 \frac{E(1+z_1)}{1-x} \left(\frac{L}{micron}\right) \left(1 - .735 \left(\frac{d}{L}\right)^2\right)^{1/2} \tag{71}$$

Furthermore, $\frac{1}{4} Bk^2 d^2 \theta^2 = k_2^2 \alpha^2$ where:

$$k_2 \equiv 1.339 \bullet 10^3 \left(\frac{d}{micron}\right) \frac{1+z_1}{1-x} E \tag{72}$$

Thus the integral in (70) can be written as:

$$Int2 = \int_0^\infty \frac{1}{k_1} erf\left(k_1 \alpha\right) \exp\left[-k_2^2 \alpha^2\right] d\alpha$$

$$= \frac{1}{k_1 k_2} \int_0^\infty erf\left(\frac{k_1 s}{k_2}\right) e^{-s^2} ds \tag{73}$$

$$= \frac{1}{k_1 k_2 \sqrt{\pi}} Arc\tan\left(\frac{k_1}{k_2}\right)$$

This quantity varies very slowly with L/d over the range 4 < L/d < 32 suggested as plausible by Aguirre. Thus to a very good approximation one obtains:

$$Int2 = 4.06 \bullet 10^{-7} \frac{(1-x)^2}{(1+z_1)^2} \left(\frac{L}{micron}\right)^{-1} \left(\frac{d}{micron}\right)^{-1} \frac{1}{E^2} \frac{1}{\sqrt{1 - .735\left(\frac{d}{L}\right)^2}} \tag{74}$$

Substituting (74) in (70), we find:

$$I = 2.3 \bullet 10^3 \Omega_d \frac{F}{1+z_2} \frac{dz_1}{\sqrt{\Omega_m (1+z_1)^3 + \Omega_\Lambda}} \left(\frac{d}{micron}\right) \frac{1}{\sqrt{1 - .735\left(\frac{d}{L}\right)^2}} \left(\frac{2Z}{A}\right)\left(\frac{\rho}{3}\right) \frac{1}{E^2} \tag{75}$$

Next we average this expression over the energy spectrum from E=.3 keV to E =8 keV. If we assume that ρ=3 and Z=A/2, this average yields:

$$\frac{dI_{halo}}{I_{core}} = 2.03 \bullet 10^3 \Omega_d \frac{dz_1}{\sqrt{\Omega_m (1+z_1)^3 + \Omega_\Lambda}} \left(\frac{d}{micron}\right) \frac{1}{\sqrt{1 - .735\left(\frac{d}{L}\right)^2}} \tag{76}$$

Aguirre has suggested that the grain distribution function in L/d might be proportional to d/L. Assuming this we can average (76) over the distribution function as follows:

$$\begin{aligned} \frac{d\langle I_{halo}\rangle}{I_{core}} &= 2.03 \bullet 10^3 \Omega_d \left(\frac{d}{micron}\right) \frac{dz_1}{\sqrt{\Omega_m (1+z_1)^3 + \Omega_\Lambda}} \frac{\int_4^{32} \frac{dx}{\sqrt{x^2 - .735}}}{\int_4^{32} \frac{dx}{x}} \\ &= 2.06 \bullet 10^3 \Omega_d \left(\frac{d}{micron}\right) \frac{dz_1}{\sqrt{\Omega_m (1+z_1)^3 + \Omega_\Lambda}} \end{aligned} \tag{77}$$

Finally we assume, as in the previous discussion concerning spherical grains, that Ω_d is constant out to a certain cutoff redshift z_c, and is zero for larger redshifts. Then,

$$\langle\tau\rangle = 2.06 \bullet 10^3 \Omega_d \left(\frac{d}{micron}\right) \int_0^{z_c} \frac{dz_1}{\sqrt{\Omega_m (1+z_1)^3 + \Omega_\Lambda}} \tag{78}$$

If Ω_m=1, Ω_Λ=0, and Ω_d=10^{-5}, (78) reduces to:

$$\langle\tau\rangle = .041\left(\frac{d}{micron}\right)\left[1 - \frac{1}{(1+z_c)^{1/2}}\right] \tag{79}$$

It is interesting to compare this result with eq'n (42), which is the analogous expression valid for spheres of the same density and composition, and where the diameter is 2 microns:

$$\frac{I_{halo}}{I_{core}} = .109\left[1 - \frac{1}{(1+z_c)^{1/2}}\right] \qquad (42)$$

For the same diameter, (79) yields an optical depth which is 75% of that for spheres. On the other hand, while (42) is linear in the sphere radius a, (79) is linear in the cylinder diameter d, and is independent of L. Aguirre has suggested that a reasonable range of grain needle diameters might be .02-.2 microns. In Fig. 9, we plot $<\tau>$ versus z_c for various d values, assuming Ω_m=.3, Ω_Λ=.7, and $\Omega_d=10^{-5}$.

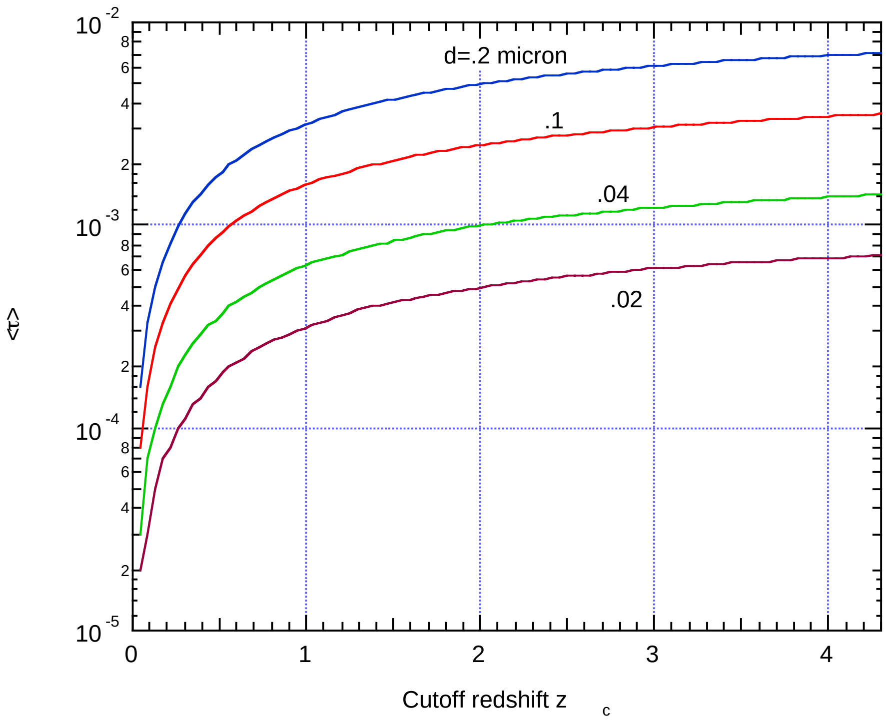


Fig. 9. Range of optical depths as calculated from (78) for various needle diameters, with Ω_m=.3, Ω_Λ=.7, and $\Omega_d=10^{-5}$.

Thedistribution of X ray intensity as a function of angle α can be calculated from eq'n (69), by integrating over z_1, E, and the L/d distribution, but not over angle α. In Fig. 10 we display this angular distribution for 2 different values of d: .02 micron (blue curves), and .2 micron (red curves). Each value of d corresponds to 4 curves: for z_c =4,3,2, and 1. It is clear from Fig. 10 that for d=.2 micron, the halo intensity is concentrated within a small angular range, whereas for d=.02 micron, the angular spread is much greater. Of course this is just what we should expect.

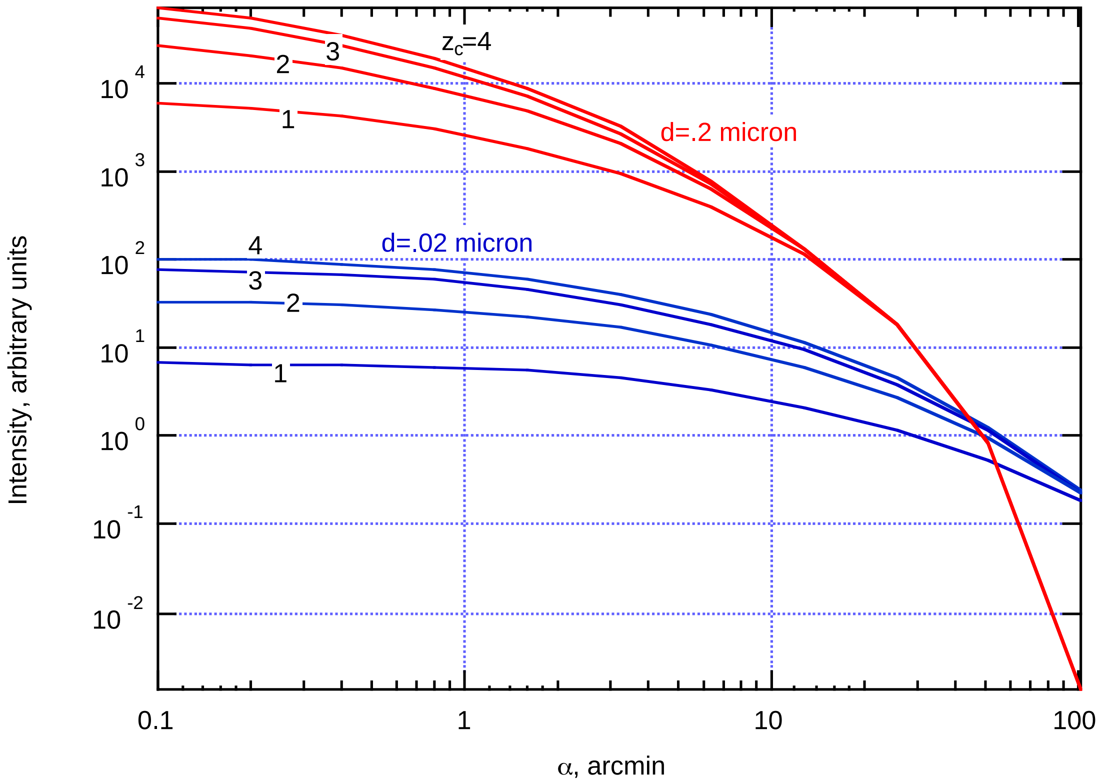


Fig. 10 Angular distribution of halo intensity for various values of d and z_c.

6. Conclusions

Fig. 9 shows that for the needle-like cylindrical grains discussed by Aguirre, the calculated optical depths are quite small. Thus, unfortunately, it appears that the observations by Paerels et al are insufficient by themselves to rule out the hypothesis of intergalactic gray dust. Evidently they would be sufficient if their sensitivity could be improved by at least an order of magnitude.