Cosmology Formulas

Eric Linder

Densities:

$$\rho_w(z) = \rho_w(0)\, e^{3\int_0^{\ln 1+z} d\ln(1+z)\,[1+w(z)]} \equiv \rho_w(0)\, f_w(z). \tag{1}$$

Hubble parameter:

$$H^2(z) = \frac{8\pi G}{3}\sum \rho_w(z) - ka^{-2} \tag{2}$$

$$= H_0^2\left[\sum \Omega_w(0)\, f_w(z) + (1-\Omega_T)\, a^{-2}\right] \tag{3}$$

where the total dimensionless density $\Omega_T = \sum \Omega_w(0)$ and the dimensionless density $\Omega_w(0) = 8\pi G\rho_w(0)/(3H_0^2)$. Note $k = (\Omega_T - 1)a_0^2 H_0^2$ is the spatial curvature constant, $a = a_0/(1+z)$, and one cannot simultaneously choose a and k dimensionless, e.g. $a_0 = 1$ *and* $k = \pm 1$.

Distances:

$$r_c(z) = \int dt/a = H_0^{-1}(1-\Omega_T)^{-1/2}\sinh[(1-\Omega_T)^{1/2}\int_0^z dz/(H/H_0)] \tag{4}$$

is the **comoving distance**. This expression is analytically valid for $\Omega_T = 0$, > 1, or < 1.

$$r_p(z) = \int dt = \int_0^z dz/[(1+z)\, H(z)] \tag{5}$$

is the **proper distance**.

Three distances based on observational methods are the angular diameter distance r_a, proper motion distance r_m, and luminosity distance r_l. In a general cosmology these are not given in terms of r_c or r_p, but within any metric theory of gravity do share the interrelationship

$$r_l = (1+z)\, r_m = (1+z)^2\, r_a. \tag{6}$$

This is closely related to the second law of thermodynamics (see Linder 1988) and so is called the thermodynamic relation. In a metric theory of gravity it is broken only if the phase space density of photons is not conserved (violation of Liouville's theorem). Note that recently the relation between r_l and r_a has been called the reciprocity relation; that in fact refers to a related relation between $r_a(z, z')$ and $r_a(z', z)$. Within general relativity the expression for any one of them comes from a second order differential equation. Within Friedmann-Robertson-Walker cosmologies, the situation simplifies considerably in that $r_m = r_c$.

Volume:

$$dV = r_a^2 \, dr_p \, d\omega \tag{7}$$

is the proper volume element, where $d\omega$ is the solid angle on the sky. The proper volume is just the integral of this. The comoving volume element $dV_c = (1+z)^3 dV$.

Note that a measurement of an angular scale θ within the plane of the sky (transverse to line of sight) probes $\theta = L/r_a$, where L is the physical scale. A measurement parallel to the line of sight probes $\Delta r = \int_x^{x+L} dr_p$. Only when the physical scale is truly infinitesimal does this reduce to $\Delta r \sim 1/[(1+z)\, H(z)]$.

Growth:

For a mass density perturbation $\delta \equiv \delta\rho/\rho$, the growth within general relativity (and where all matter and only matter is perturbed) is governed by a second order differential equation

$$\ddot{\delta} + 2H\dot{\delta} - (3/2)\Omega_m(a) H^2 \delta \;=\; 0 \tag{8}$$

$$\delta'' + (2-q)a^{-1}\delta' - (3/2)\Omega_m(a)\, a^{-2}\delta \;=\; 0 \tag{9}$$

where dot denotes a time derivative and prime a derivative with respect to scale factor a. The deceleration parameter $q = -a\ddot{a}/\dot{a}^2 = -1 - d\ln H/d\ln a$.

The (growing) solution for a pure matter universe is $\delta \sim a$, and observations of large scale structure indicate that our universe must have been substantially matter dominated for $z \approx 2 - 1000$, so it is convenient to define a normalized growth variable $g \equiv \delta/a$. The equation for this matter density growth variable, when all matter and only matter is perturbed, is

$$g'' + \left[4 + \frac{1}{2}(\ln H^2)'\right] g' + \left[3 + \frac{1}{2}(\ln H^2)' - \frac{3}{2}\Omega_m(a)\right] g = 0 \tag{10}$$

Here a prime is derivative with respect to $\ln a$. The matter density $\Omega_m(a) = \Omega_m a^{-3}/[H/H_0]^2$.