Geodesic Equation

A straight line in locally inertial coordinates, $d^2\xi^\alpha/d\tau^2 = 0$, with $d\tau^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$, gives rise to an equation of motion in lab coordinates x^μ of

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{1}$$

The physical quantity of the "path of motion of a particle" must be the same whether it is described as "a straight line in a local Lorentz frame" or as "a geodesic of the spacetime geometry".

a) Show that you obtain equation (1) by calculating the geodesic of the metric, i.e. the extremization of the proper time $d\tau = (g_{\mu\nu}\, dx^\mu dx^\nu)^{1/2}$ along a path between two fixed spacetime points:

$$\delta \int_A^B d\sigma \left[g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} \right]^{1/2} = 0.$$

Hint: Relate $\delta g_{\mu\nu} = (\partial g_{\mu\nu}/\partial x^\lambda)\, \delta x^\lambda$ to $\Gamma^\lambda{}_{\mu\nu}$. Integrate by parts and remember the endpoint variations vanish.

b) Using (1), verify that the equations of motion in Cartesian coordinates for a particle moving in Minkowski space give a straight line.

c) Using (1), verify that the equations of motion in spherical coordinates for a particle moving in Minkowski space give a straight line.

Hint: recall the line element is $d\tau^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ and the Christoffel symbols are not all zero. It simplifies the algebra of the geodesic equations to consider a fixed plane (why is this legitimate?).

Hint: it is probably easiest to see the motion is in a straight line by considering $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, etc. It is also possible to use angular momentum conservation (from the angular geodesic equations) to derive $d^3(r^2)/dt^3 = 0$, implying r^2 is a second order polynomial, and then show this gives straight line uniform motion.

d) Write down the geodesic equations for the Schwarzschild metric. Does initially radial motion stay radial?