Relativistic Doppler Shift

You are familiar with the derivation of the Doppler shift using Galilean transformations. Here you will use Lorentz transforms.

a) Consider a source at rest with respect to the observer, i.e. with four-velocity $u^\alpha = (1, \vec{0})$. Now consider a moving source, i.e. one in a Lorentz boosted frame. What is the source four-velocity?

b) Energy is a scalar, so the energy of a particle measured by an observer is the projected four momentum $E = -\mathbf{p} \cdot \mathbf{u} = -p^\alpha u_\alpha$, where p^α is the particle four-momentum in some frame and u^α is the four-velocity (boost) of that frame with respect to the observer. Check this for a frame at rest with respect to the observer by calculating it for a massive particle 1) at rest, 2) with velocity $\vec{v}$. The results should look familiar.

c) Now allow the frame to move with velocity $\vec{v}$ and consider a photon of energy E_e emitted from a source at rest in that frame (i.e. the source moves with velocity $\vec{v}$ with respect to the observer). Relate E_e to the energy E_o the observer measures. As $v \to 0$ you should find the usual Doppler shift formula. Define a new parameter $z = E_e/E_o - 1$. Is z positive or negative for a source receding (radially) from the observer? approaching (radially) the observer? At what angle between the source velocity and the line of sight to the observer does z change from negative to positive? In the Galilean case this angle is 90° – there is no transverse Doppler shift. In the Lorentz transverse case what is the shift to lowest order in v?

Since for a photon the energy is inversely proportional to the wavelength, this effect (and z) is sometimes called the redshift or blueshift. The scalar product of four-vectors above will also hold in general relativity and exhibits the equivalence of the photon energy shift from a moving source, the gravitational redshift of a black hole, and the shift of a time dependent gravitational lens. (You will soon learn that this is an aspect of the Principle of Equivalence underlying general relativity).