Recombination

Recombination ($p + e \to H + \gamma$) occurs when the ionization fraction $X = n_e/n_b = n_e/(n_p + n_H)$ of the universe drops much below unity. X is given by the Saha formula

$$\frac{1-X}{X^2} = [4\zeta(3)\sqrt{2/\pi}](T/m_e)^{3/2}\eta\, e^{Q/T}$$

where the baryon-photon ratio $\eta = n_b/n_\gamma$ and the hydrogen binding energy $Q = m_p + m_e - m_H$.

a) From the phase space integral for the number density $n = g\,(2\pi)^{-3}\int d^3p\, f(p)$, where $p^2 = E^2 - m^2$ and $f(p) = [e^{(E-\mu)/T} \pm 1]^{-1}$, show that in the relativistic limit $m/T \ll 1$, $\mu/T \ll 1$, the density for a boson (e.g. photons: $g = 2$; chemical potential $\mu = 0$) is

$$n = \pi^{-2}T^3 \int_0^\infty dx\, x^2(e^x - 1)^{-1} \equiv 2\pi^{-2}\zeta(3)T^3$$

b) Redo the integral for nonrelativistic ($m/T \gg 1$) fermions with chemical potential μ and derive the Boltzmann formula

$$n = g\,(mT/2\pi)^{3/2}e^{-(m-\mu)/T}.$$

c) Derive the Saha formula for the ionization fraction given above.

Hint: Use the definition of X in the form $X = n_e/n_b = n_e\,(n_\gamma/n_b)\,n_\gamma^{-1}$. Then use the conservation of chemical potential $\mu_p + \mu_e = \mu_H$ and substitute in $\mu_p(n_p)$, $\mu_H(n_H)$. Finally, use charge conservation $n_e = n_p$.
Hint: Protons and electrons are spin-1/2 particles; what are their g factors? A hydrogen atom is made up of one proton and one electron; what is its g factor? Also, it is a good approximation to take $m_p = m_H$ everywhere but in the exponential argument.